ON THE GEOMETRIC ARENA OF SUPERMECHANICS

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Abstract

In the case of simple graded manifolds utilized in supermechanics, supervector fields and exterior superforms are represented by global sections of smooth vector bundles.

A BRST extension of Hamiltonian mechanics [4, 6] shows that (i) one should consider vector bundles in order to introduce generators of BRST and anti-BRST transformations, and (ii) one can narrow the class of superfunctions under consideration because the BRST extension of a Hamiltonian is a polynomial in odd variables.

The main ingredient in a theory of supermanifolds is the sheaf $\mathcal{B}$ of graded commutative algebras on a manifold Z. If this sheaf fulfills the Rothstein axioms, it is said to be an R-supermanifold [1, 3]. In particular, the notion of an R-supermanifold includes graded manifolds, supermanifolds of A.Rogers and infinite-dimensional supermanifolds of A.Jadczyk and K.Pilch. This notion also implies that the sheaf $\mathrm{Der}\,\mathcal{B}$ of graded differentiations of $\mathcal{B}$ and the dual sheaf $\mathrm{Der}^*\mathcal{B}$ are introduced. Let U be an open subset of Z and $\mathcal{B}\mid_U$ the restriction of the sheaf $\mathcal{B}$ to U. By a graded differentiation of the sheaf $\mathcal{B}\mid_U$ is meant its endomorphism u such that

$$u(ff') = u(f)f' + (-1)^{[u][f]} f u(f')$$

for the homogeneous elements $u \in \mathrm{Der}\,\mathcal{B}$ and $f, f' \in \mathcal{B}\mid_U$. We will use the notation $[.]$ of the Grassman parity. The graded differentiations of $\mathcal{B}\mid_U$ constitute the $\mathcal{B}$-module $\mathrm{Der}\,\mathcal{B}(\mathrm{U})$, and the presheaf of these $\mathcal{B}$-modules generates the sheaf $\mathrm{Der}\,\mathcal{B}$.

Its elements are called supervector fields on a manifold Z. The dual of the sheaf $\mathrm{Der}\,\mathcal{B}$ is the sheaf $\mathrm{Der}^*\mathcal{B}$ generated by the $\mathcal{B}$-linear morphisms

$$\phi : \mathrm{Der}\mathcal{B}(\mathrm{U}) \to \mathcal{B}_{\mathrm{U}}. \tag{1}$$

One can think of its elements as being 1-superforms on a manifold Z.

We will consider the following class of graded manifolds, called simple graded manifolds [2]. Given a vector bundle $E \to Z$ with an m-dimensional typical fiber V, let us consider the fiber bundle

$$\wedge E^* = \mathbf{R} \underset{Z}{\oplus} (\overset{m}{\underset{k=1}{\oplus}} \overset{k}{\wedge} E^*) \tag{2}$$

whose typical fiber is the finite Grassman algebra $\Lambda^* = \wedge V^*$. Note that there is a different notion of a Grassman algebra [5], which is not equivalent to the above one if V is finite-dimensional [3]. Global sections of $\wedge E^* \to Z$ (2), called superfunctions, make up a $\mathbf{Z}_2$-graded ring $\mathcal{B}^0$. Let $\{c^a\}$ be the holonomic bases in $E^* \to Z$ with respect to some bundle atlas with transition functions $\{\rho^a_b\}$, i.e., $c'^a = \rho^a_b(z)c^b$. Then superfunctions read

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1\ldots a_k} c^{a_1} \cdots c^{a_k}, \tag{3}$$

where $f_{a_1 \cdots a_k}$ are local functions on Z, and we omit the symbol of exterior product of elements c. The coordinate transformation law of superfunctions (3) is obvious. The sheaf of these superfunctions belongs to the above mentioned class of simple graded manifolds. In this case, supervector fields and exterior superforms on Z can be represented by sections of smooth vector bundles over Z as follows.

Since the canonical splitting $VE = E \times E$, the vertical tangent bundle $VE \to E$ can be provided with the fiber bases $\{\partial_a\}$ dual of $\{c^a\}$. These are fiber bases of $\mathrm{pr}_2 VE = E$. Let (z^A) be coordinates on Z. Then supervector fields on a manifold Z read

$$u = u^A \partial_A + u^a \partial_a \tag{4}$$

where u^A, u^a are local superfunctions (3). A supervector field u (4) is homogeneous iff

$$[u^A] = [u^B] = [u^a] + 1 = [u^b] + 1 = [u], \qquad \forall A, B, a, b.$$

It defines the graded endomorphism of $\mathcal{B}^0$ by the law

$$u(f_{a\ldots} c^a \cdots) = u^A \partial_A (f_{a\ldots}) c^a \cdots + u^a f_{a\ldots} \partial_a \rfloor (c^a \cdots). \tag{5}$$

A direct computation shows that this law implies the corresponding coordinate transformation law

$$u'^A = u^A, \qquad u^a = \rho^a_j u^j + u^A \partial_A(\rho^a_j) c^j$$

of supervector fields. It follows that supervector fields can be represented by sections of the vector bundle $\mathcal{P}_E \to Z$ which is locally isomorphic to the vector bundle

$$\mathcal{P}_E \mid_U \approx \wedge E^* \underset{Z}{\otimes} (\mathrm{pr}_2 VE \underset{Z}{\oplus} TZ) \mid_U,$$

and has the transition functions

$$\begin{aligned} z'^A_{i_1\ldots i_k} &= \rho^{-1}{}^{a_1}_{i_1} \cdots \rho^{-1}{}^{a_k}_{i_k} z^A_{a_1\ldots a_k}, \\ v'^i_{j_1\ldots j_k} &= \rho^{-1}{}^{b_1}_{j_1} \cdots \rho^{-1}{}^{b_k}_{j_k} \left[\rho^i_j v^j_{b_1\ldots b_k} + \frac{k!}{(k-1)!} z^A_{b_1\ldots b_{k-1}} \partial_A(\rho^i_{b_k}) \right] \end{aligned}$$

of the bundle coordinates $(z^A_{a_1\ldots a_k}, v^i_{b_1\ldots b_k})$, $k = 0, \ldots, m$. These transition functions fulfill the cocycle relations. There is the exact sequence over Z of vector bundles

$$0 \to \wedge E^* \underset{Z}{\otimes} \mathrm{pr}_2 VE \to \mathcal{P}_E \to \wedge E^* \underset{Z}{\otimes} TZ \to 0.$$

It is readily observed that every linear connection

$$\Gamma = dz^A \otimes (\partial_A + \Gamma_A{}^a{}_b v^b \partial_a) \tag{6}$$

on the vector bundle $E \to Z$ yields the splitting

$$\Gamma_S : u^A \partial_A \mapsto u^A(\partial_A + \Gamma_A{}^a{}_b c^b) \tag{7}$$

of this exact sequence and the corresponding decomposition

$$u = u^A \partial_A + u^a \partial_a = u^A(\partial_A + \Gamma_A{}^a{}_b c^b \partial_a) + (u^a - u^A \Gamma_A{}^a{}_b c^b)\partial_a$$

of sections of $\mathcal{P}_E$. One can think of Γ_S as being a superconnection, but this is not a connection on the fiber bundle $\mathcal{P}_E \to Z$ in the conventional sense. The sheaf of sections of $\mathcal{P}_E \to Z$ is isomorphic to the sheaf $\mathrm{Der}\,\mathcal{B}$. Global sections of the vector bundle $\mathcal{P}_E \to Z$ constitute the $\mathcal{B}^0$-module of supervector fields on Z which is also a Lie superalgebra with respect to the bracket

$$[u, u'] = uu' + (-1)^{[u][u']+1} u'u.$$

Similarly, the $\mathcal{B}^0$-dual $\mathcal{P}_E^*$ of $\mathcal{P}_E$ is a vector bundle over Z which is locally isomorphic to the vector bundle

$$\mathcal{P}_E^* \mid_U \approx \wedge E^* \underset{Z}{\otimes} (\mathrm{pr}_2 V E^* \underset{Z}{\oplus} T^* Z) \mid_U,$$

and has the transition functions

$$\begin{aligned}
v'_{j_1\ldots j_k j} &= \rho^{-1}{}^{a_1}_{j_1} \cdots \rho^{-1}{}^{a_k}_{j_k} \rho^{-1}{}^{a}_{j} v_{a_1\ldots a_k a}, \\
z'_{i_1\ldots i_k A} &= \rho^{-1}{}^{b_1}_{i_1} \cdots \rho^{-1}{}^{b_k}_{i_k} \left[z_{b_1\ldots b_k A} + \frac{k!}{(k-1)!} v_{b_1\ldots b_k j} \partial_A (\rho^j_{b_k}) \right]
\end{aligned}$$

of the bundle coordinates $(z_{a_1\ldots a_k A}, v_{b_1\ldots b_k j})$, $k = 0, \ldots, m$, with respect to the dual bases $\{dz^A\}$ in T^*Z and $\{dc^b\}$ in $\mathrm{pr}_2 V^* E = E^*$. There is the exact sequence

$$0 \to \wedge E^* \underset{Z}{\otimes} T^* Z \to \mathcal{P}_E^* \to \wedge E^* \underset{Z}{\otimes} \mathrm{pr}_2 V E^* \to 0. \tag{8}$$

The sheaf of sections of $\mathcal{P}_E^* \to Z$ is isomorphic to the sheaf $\mathrm{Der}^* \mathcal{B}$. Global sections of the vector bundle $\mathcal{P}^* \to Z$ constitute the $\mathcal{B}^0$-module of exterior 1-superforms

$$\phi = \phi_A dz^A + \phi_a dc^a \tag{9}$$

on Z with the coordinate transformation law

$$\phi'_a = \rho^{-1}{}^b_a \phi_b, \qquad \phi'_A = \phi_A + \rho^{-1}{}^b_a \partial_A (\rho^a_j) \phi_b c^j.$$

The superform (9) is homogeneous iff

$$[\phi_A] = [\phi_B] = [\phi_a] + 1 = [\phi_b] + 1 = [\phi], \qquad \forall A, B, a, b.$$

Then the morphism (1) can be seen as the interior product

$$u \rfloor \phi = u^A \phi_A + (-1)^{[\phi_a]} u^a \phi_a. \tag{10}$$

Similarly to (7), every linear connection Γ (6) on the vector bundle $E \to Z$ yields the splitting of the exact sequence (8) and the corresponding decomposition

$$\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \Gamma_A{}^a{}_b \phi_a c^b) dz^A + \phi_a (dc^a - \Gamma_A{}^a{}_b c^b dz^A)$$

of 1-superforms on Z.

As in the non-graded case, by exterior k-superforms ϕ are meant sections of the graded exterior product $\overline{\wedge}^k_Z \mathcal{P}^*_E$ which is the quotient of the tensor product $\otimes^k_Z \mathcal{P}^*_E$ with respect to its subbundle generated by elements

$$\varphi \otimes \varphi' + (-1)^{[\varphi][\varphi']} \varphi' \otimes \varphi, \qquad \varphi, \varphi' \in \mathcal{P}^*_E.$$

It is readily observed that k-superforms, $k = 0, 1, \ldots$, constitute a $(\mathbf{Z}, \mathbf{Z}_2)$-bi-graded ring $\mathcal{B}^*$ with respect to the graded exterior product such that

$$\phi \overline{\wedge} \sigma = (-1)^{|\phi||\sigma| + [\phi][\sigma]} \sigma \overline{\wedge} \phi.$$

The interior product (10) is extended to the ring $\mathcal{B}^*$ by the rule

$$u \rfloor (\phi \overline{\wedge} \sigma) = (u \rfloor \phi) \overline{\wedge} \sigma + (-1)^{|\phi| + [\phi][u]} \phi \overline{\wedge} (u \rfloor \sigma).$$

Using (5), one can introduce the graded exterior differential of superfunctions in accordance with the condition

$$u \rfloor df = u(f) \tag{11}$$

for an arbitrary supervector field u (this condition differs from that of [1]). The graded differential is extended uniquely to the ring $\mathcal{B}^*$ of superforms by the familiar rules

$$d(\phi \overline{\wedge} \sigma) = (d\phi) \overline{\wedge} \sigma + (-1)^{|\phi|} \phi \overline{\wedge} (d\sigma), \qquad dd = 0.$$

It takes the coordinate form

$$d\phi = dz^A \overline{\wedge} \partial_A(\phi) + dc^a \overline{\wedge} \partial_a(\phi),$$

where the left derivatives ∂_A, ∂_a act on the coefficients of superforms by the rule (5), and are graded commutative with the forms dz^A, dc^a. With the interior product and the exterior graded differential, the Lie derivative of a superform ϕ along a supervector field u can be given by the familiar formula

$$\mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi). \tag{12}$$

If f is a superfunction, we obtain from (11) and (12) that $\mathbf{L}_u f = u(f)$ as usual.

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